On the Structure of Connected Compact Sets in Harmonic Space

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1 Preliminary definitions

Definition 1.1 (frequency ratio)**.** A frequency ratio fr is a ratio n/d that represents the relative frequency of a tone in comparison to a unison $(1/1)$ such that n and d are coprime.

Definition 1.2 (collapsed frequency ratio)**.** a collapsed frequency ratio $c(fr)$ is a frequency ratio with the additional property that $1 \leq n/d < 2$. A frequency ratio can be collapsed by multiplying it successively by $2/1$ if $n/d < 1$ or by $1/2$ if $n/d \geq 2$ until $1 \leq n/d < 2$.

Definition 1.3 (Harmonic Space)**.** Harmonic Space HS_l is a multi-dimensional lattice where each dimension represents a unique prime factor up to the limit l (or any finite set of primes). Each point represents a frequency ratio based on the factorization of n and d such that the p-adic valuation $v_p(n)$ and $v_p(d)$ is equivalent to the number of positive and negative steps, respectively, in each dimension p outward from a reference point (e.g. $1/1$) on the lattice.

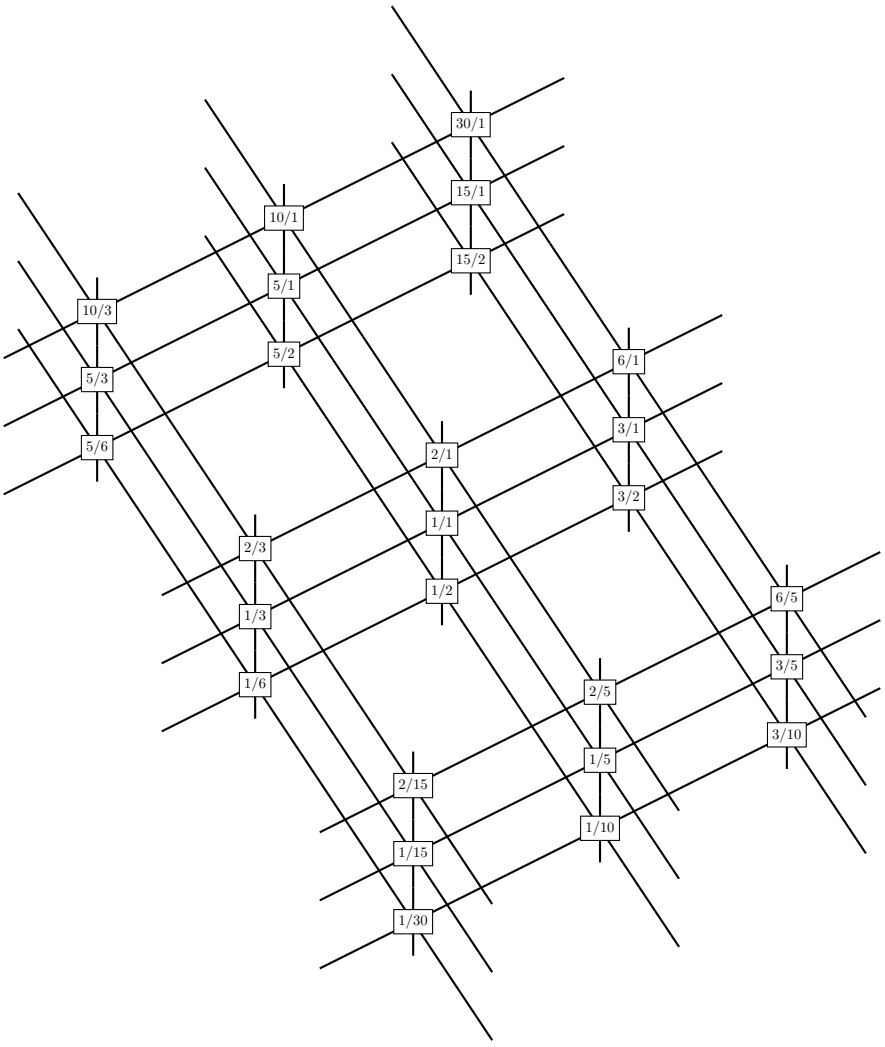


Figure 1: Example of a subset of HS_5

Definition 1.4 (Collapsed Harmonic Space)**.** A Collapsed Harmonic Space CHS_l is the same as Harmonic Space except the dimension representing movement by the prime 2 is omitted and each point is collapsed.

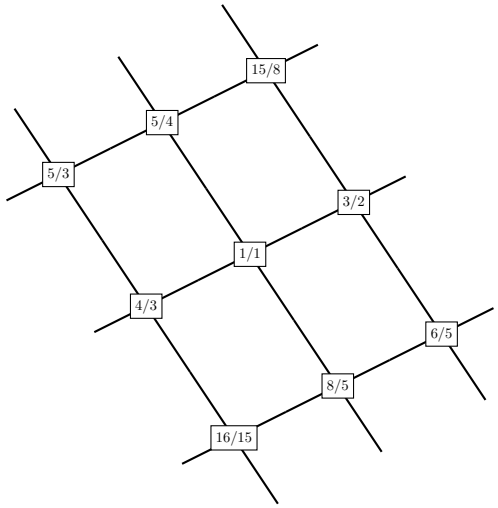


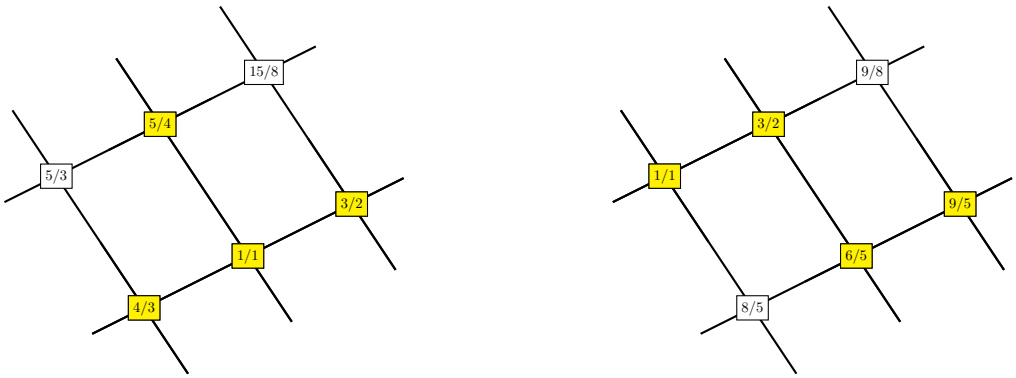
Figure 2: Example of a subset of CHS_5

Definition 1.5 (Compact Set)**.** A Compact Set in (Collapsed) Harmonic Space CS is a set of points (a chord) $((n/d)_1, (n/d)_2, ...(n/d)_k)$ that form a connected sublattice of k-points in Harmonic Space. This means that for each point in a Compact Set, there exists another point in the Set that has the same p-adic valuations of the numerator and denomenator $v_p(n)$ and $v_p(d)$ for all primes except a single prime which differs by only 1:

$$\sum_{p=2}^{l} |(v_p(n_i) + v_p(d_i)) - (v_p(n_j) + v_p(d_j))| = 1 \tag{1}$$

Similarly, in Collapsed Harmonic Space:

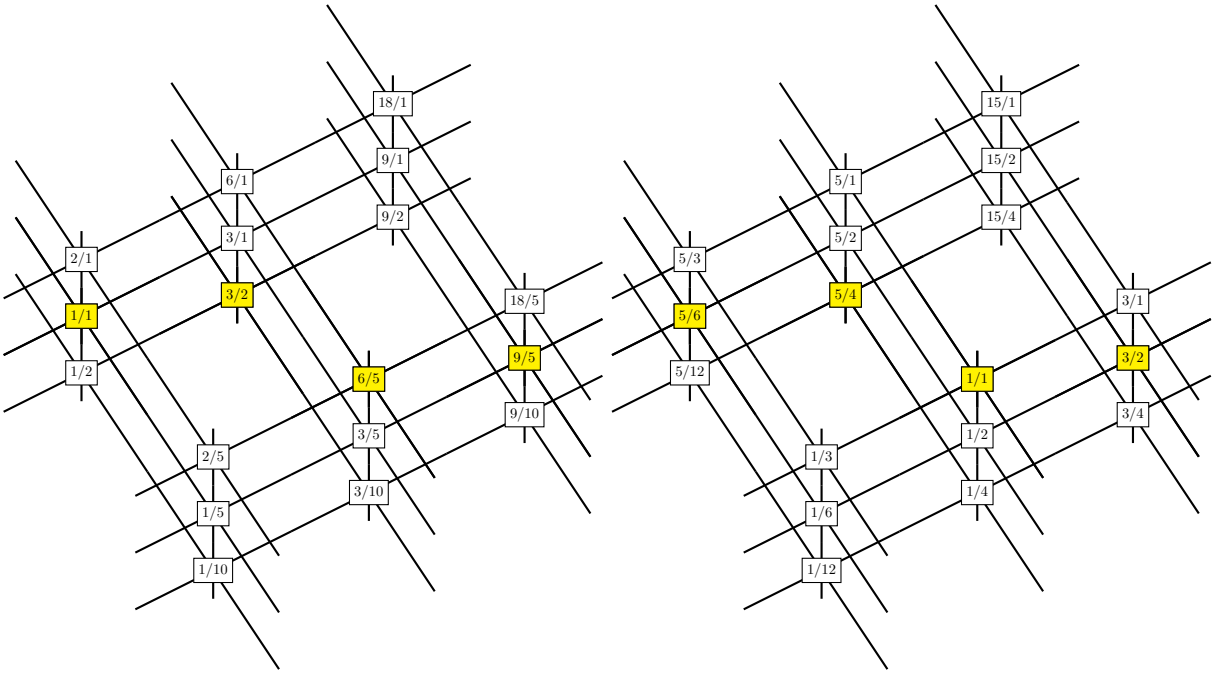
$$\sum_{p>2}^{l} |(v_p(n_i) + v_p(d_i)) - (v_p(n_j) + v_p(d_j))| = 1 \tag{2}$$



(a) (b)

Figure 3: Two examples of Compact Sets of size 4 (highlighted in yellow) in CHS_5

Definition 1.6 (Transposition)**.** A Transposition of a Compact Set $T_{fr}(CS)$ is the multiplication of all elements of the set by a frequency ratio. Note that a transposition is only well defined in a non-collapsed Harmonic Space. As such, a transposition in Collapsed Harmonic Space requires the Compact set to be mapped into non-collapsed Harmonic Space.



(a) (b)

Figure 4: Example of Compact Set from Fig. 3b expanded into Non-Collapsed Harmonic Space (a) and then transposed (b) by a frequency ratio of 5/6: minus one step in the prime 2 dimension, minus one step in the prime 3 dimension, and plus one step in the prime 5 dimension

2 Graph of the Structure of Compact Sets in Collapsed Harmonic Space

Let $S_{j,k}(CHS_l))$ be the set of all Compact Sets of size j to k in Collapsed Harmonic Space CHS_l such that at least one point is equal to 1/1. This represents all possible chords of CHS_l.

We define a graph $G_{a,b}(S_{j,k}(CHS_l))$ were the vertices represent the elements of $S_{j,k}(CHS_l)$ and an edge is induced between two vertices, v_1 and v_2, if the size of the symmetric difference of v_1 and some transposition of v_2 (i.e., $T_{fr}(v_2)$) is between a and b: i.e., $a \leq |v_1 \Delta T_{fr}(v_2)| \leq b$. For example, if $a = 1$ and $b = 2$, all but one note stays the same from chord-to-chord if they are the same size and only one note is added or removed if the chords are of different sizes. Note that these generalizations came from an original investigation where $j = k$ and $a = b = 2$, which is all Compact Sets of size j such that from chord-to-chord, only one note changes.

One could also remove or weight edges based on whether or not the movement of the fr that changes between the two compact sets is within a certain melodic threshold (i.e., moves by less than a certain interval in pitch space). Let $v_1 \Delta T_{fr}(v_2) = (fr_1, fr_2)$, then keep or weight edge if $1200(\log_2(fr_1) - \log_2(fr_2))$ is less than an arbitrary limit (e.g. 200 for a whole tone).

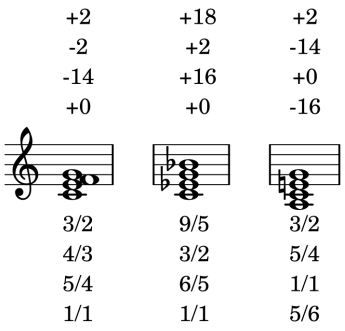


Figure 5: Chords in music notation (with frequency ratios below and cent-deviations above) of Fig. 3a, 3b (equivalent to 4a), and 4b, respectively, demonstrating that Fig. 3a and 3b only satisfy the edge requirement in $G_{2,2}(S_{4,4}(CHS_5))$ under transposition by a frequency ratio of 5/6

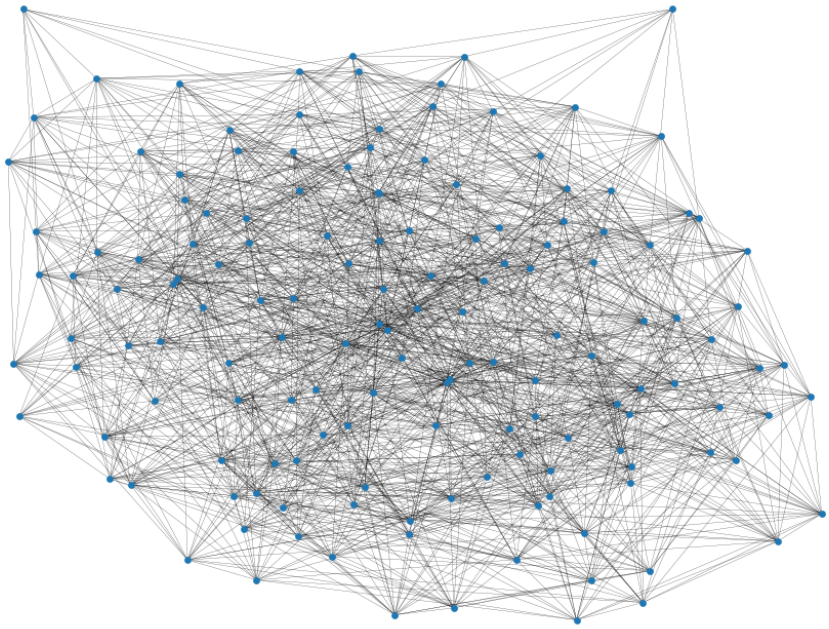


Figure 6: Example of graph $G_{2,2}(S_{4,4}(CHS_5))$

3 Open Questions in Graph Theory and Computational Complexity

1. Are these graphs Hamiltonian and if so, under what conditions?
2. Is there an efficient way to build/generate all compact sets?
3. Given two compact sets, is there an efficient way to determine if a transposition exists such that the size of the symmetric difference satisfies $a \leq |CS_1 \Delta T_{fr}(CS_2)| \leq b$ for a given a and b?

4 Notes on representation

Programatically, I have represented each fr as a set of primes (positive for the numerator and negative for the denominator): $(3, -5) = 3/5$ in HS and $6/5$ in CHS. An example compact set might be: $((), (-3), (3), (3, -5))$ where $()$ equals $1/1$. Within a frequency ratio, there can be several instances of each prime, but each instance must be either positive or negative: $(3, 3, 5)$ or $(-3, -3, 5)$ but not $(-3, 3, 5)$ as -3 and 3 would cancel each other out in the resulting frequency ratio. An alternative would be to store the p-adic valuation for each prime: $(-3, -3, 5)$ would be equivalent to $[-2, 1]$ or $(-3, -3, 5, 5, 7, 13)$ would be equivalent to $[-2, 2, 1, 0, 1]$. Both representations provides the coordinates in Harmonic Space such that the resulting frequency ratio can easily be computed.