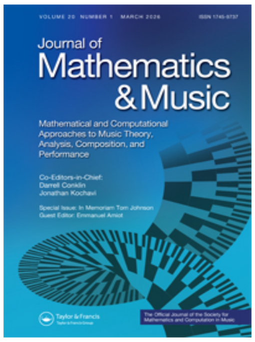

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Journal of Mathematics and Music

Mathematical and Computational Approaches to Music Theory, Analysis, Composition and Performance

ISSN: 1745-9737 (Print) 1745-9745 (Online) Journal homepage: www.tandfonline.com/journals/tmam20

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To cite this article: Michael Winter (2026) Tom Johnson and a rational theory of harmony, Journal of Mathematics and Music, 20:1, 77-93, DOI: 10.1080/17459737.2025.2591899

To link to this article: https://doi.org/10.1080/17459737.2025.2591899

Published online: 28 Jan 2026.

Journal of Mathematics and Music, 2026
Vol. 20, No. 1, 77–93, https://doi.org/10.1080/17459737.2025.2591899

Tom Johnson and a rational theory of harmony

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(*Received 18 April 2025; accepted 14 November 2025*)

In this article, I present a theory of harmony on *the structure of conjunct connected sets in harmonic space*. The theory is a product of the combined influence of Tom Johnson, James Tenney, and Larry Polansky. The title of the article is a respectful nod to James Tenney's seminal text "John Cage and the theory of harmony" which demonstrates how the ideas of Cage, who often remarked that his music had nothing to do with harmony, actually could serve as the basis for a new theory of harmony. Similarly, in this article, I aim to show how the ideas of Johnson, who claimed that he did not have much interest in tuning beyond equal-temperament, can form the basis of a rational theory of harmony.

Keywords: Harmony; harmonic space; harmonic replacement; connected subgraphs; graph theory; Hamiltonicity; musical morphologies

2020 Mathematics Subject Classifications: 05C45; 03D15; 05C90; 00A65

1. Introduction: discreteness in the work of Tom Johnson

Tom Johnson's mathemusical ideas were almost exclusively discrete. He did not implement continuous mathematics. Further, while he used the word "rational" in terms of "rational logic", as suggested by the title of his work *Rational Melodies*, he never used rational numbers to express pitches. Johnson felt that equal-temperament afforded a simple quantization and subsequent mapping onto the integers through which he could explore his mathemusical ideas despite the fact that pitches in equal-temperament, while integers in the pitch domain, are irrational in the frequency domain.

Johnson was, however, deeply concerned with harmony. This is made evident by the title and content of his book *Other Harmony* (Johnson 2014). The book also demonstrates just how well suited the integer domain was for Johnson's harmonic ideas. He had little interest in just-intonation, in which the relationship between pitches are expressed as whole-number frequency ratios. I believe this is because musical scales in just-intonation are untempered and acyclic. They do not favor the accuracy of one interval over another such as with different well- and equal-temperaments, which prioritize and sacrifice the accuracy of different intervals for key cyclicness. Well-temperaments and eventually the recent standard of equal temperament, which Johnson adopted, are workarounds to what Larry Polansky called "the historical tuning problem" (Polansky et al. 2009) which results from the fact that a prime raised to a power will never equal a different prime raised to any power: $a^m \neq b^n$ for any prime numbers a and b and any positive integers m and n.

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This article has been corrected with minor changes. These changes do not impact the academic content of the article.

Despite the fact that Johnson did not explore rational tunings systems, I would like to take this opportunity to demonstrate how applicable Johnson's ideas are to just-intonation. In particular, I will be connecting Johnson's graph theoretical ideas to James Tenney's theory of *harmonic distance in harmonic space* and Larry Polansky's methods of *harmonic replacement*. I will proceed by starting with a formalization of Johnson's graph theoretical ideas and an overview of the concepts of harmonic space and harmonic replacement. I will then give a new theory of mine, *on the structure of conjunct connected sets in harmonic space*, which would only have arisen based on my influence from Johnson, Tenney and Polansky combined. I conclude with musical examples and open mathemusical questions that have resulted from this theory.

The title of this article is a respectful nod to James Tenney's seminal article "John Cage and the theory of harmony" (Tenney 2015a) which demonstrates how the ideas of Cage, who often remarked that his music had nothing to do with harmony, actually could serve as the basis for a new theory of harmony. Similarly, in this article, I aim to show how the ideas of Johnson, who claimed that he did not have much interest in tuning beyond equal-temperament, can form the basis of a rational theory of harmony.

2. Musical graphs and morphological constraints

One of Johnson's most important mathemusical ideas was his creation and use of *musical graphs* in which vertices and edges comprise musical elements and relationships between them, respectively. This type of graph-theoretical thinking is featured prominently in Johnson's compositional drawings, which then became the primary subject of his book *Looking at Numbers* (Johnson and Jedrzejewski 2014). In this section, I would like to give an overview of a formalization of this methodology that I detailed in my article "Minimal change musical morphologies" (Winter 2017).

A musical graph can be derived from a set of *morphological constraints* at different hierarchical levels. At the atomic level are *combinatorial constraints*. These constraints determine the musical elements, which become the vertices of a graph. At the next higher hierarchical level are *local morphological constraints*. These are relationships between the elements defined by the combinatorial constraints, which become the edges of a graph. To generate a musical morphology, Johnson would often find a path on such a graph using a *global morphological constraint*, which is a statistical property of the path (e.g. a Hamiltonian path in which every vertex is traversed once and only once).

My canonical example of this paradigm is Johnson's *Trio*. In this piece for three stringed instruments, each pitch in a four-octave chromatic set is indexed by a number between 0 and 48 where middle C equals 24. The musical morphology enumerates through all three-note chords satisfying the combinatorial constraint that the numbers representing the pitches within each chord are distinct integer partitions without repetition of 72, e.g. $13 + 24 + 35$. The local morphological constraint is that from chord to chord, one pitch must remain the same while the other pitches move by a semitone in contrary motion. The global morphological constraint is that each chord occurs only once. Johnson's compositional drawing for the piece and an excerpt of the score are provided in Figures 1 and 2, respectively. Johnson implemented this compositional paradigm in a myriad of other works such as *Falling Thirds with Drum*, *Mocking*, and *Block Design for Piano*.

There are several aspects of Johnson's *Trio* that are particularly relevant for the theory I will outline below. (1) The enumeration through the set of chords is exhaustive yet as short as possible in that each chord sounds once and only once. (2) The piece is ultimately about voice leading using contrapuntal motion.

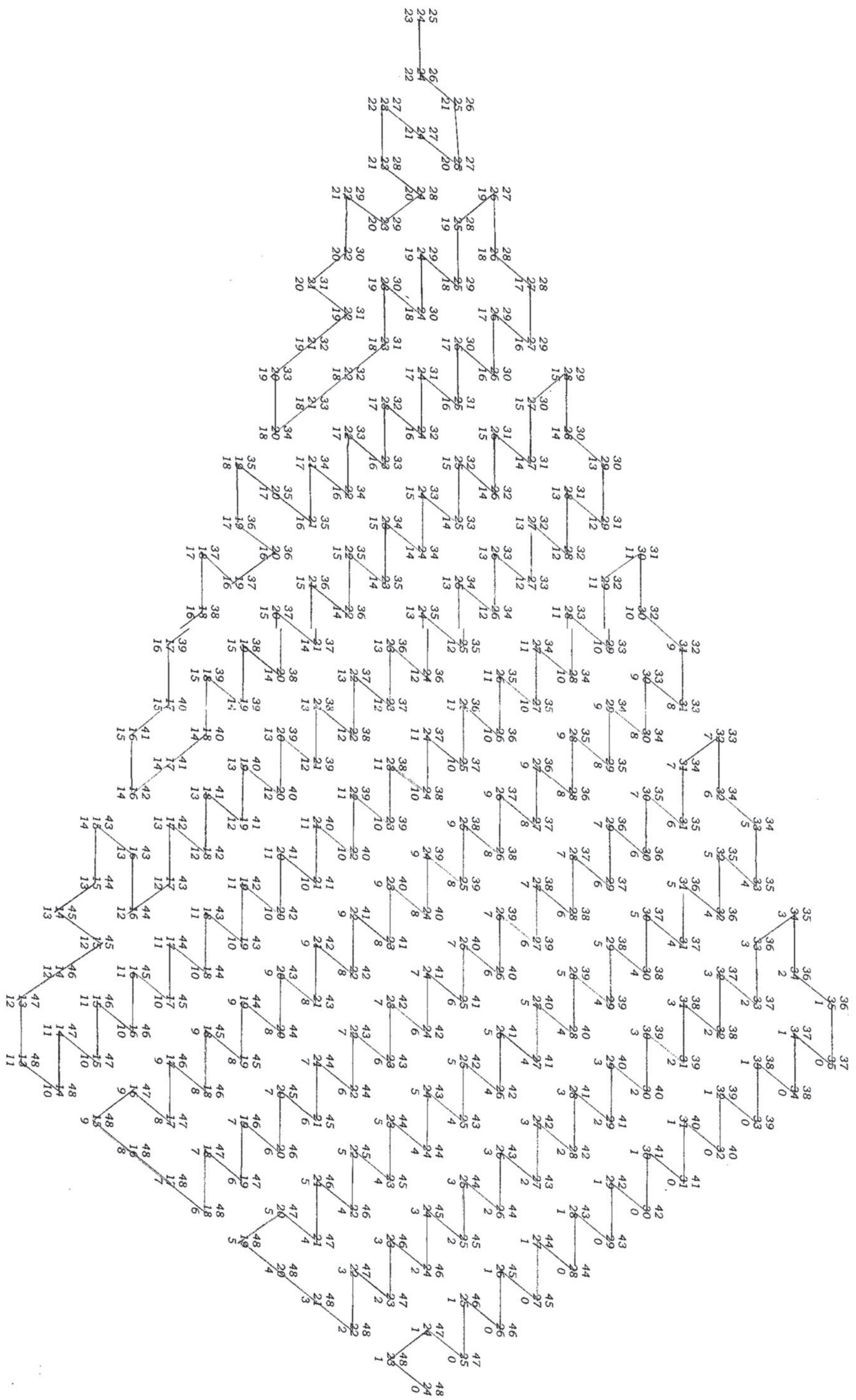


Figure 1. *Trio* graph (from cover of score). Note that this figure shows the Hamiltonian path used by Johnson for the composition, but not all edges that satisfy the local morphological constraint.

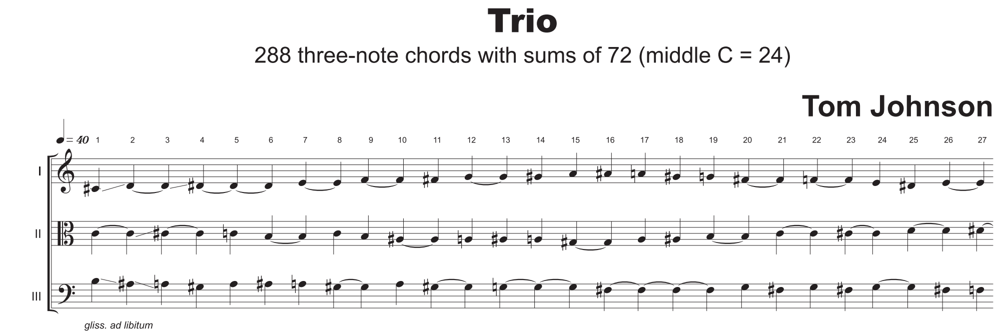


Figure 2. First system of *Trio*.

3. Harmonic distance in harmonic space

The formalization of James Tenney's concept of *harmonic distance* in *harmonic space* was motivated by an aim to quantify the consonance/dissonance of musical intervals, a subject that has long captivated both musicians and mathematicians. Numerous prior attempts have yielded various formulations, many of which share a common feature: their basis on the complexity of the numbers used to express the frequency ratios between tones. More specifically, these formulations often depend on the difficulty of factoring the numerator and denominator of a frequency ratio, accounting for the number, size, and exponents of their prime factors. I often explain the quantification of consonance/dissonance in terms of computational complexity in that it relates to the amount of time needed to compute the prime factors of a number. That is, two pitches are perceived as being more closely related because it takes a relatively short time for the brain to factor the numerator and denominator of the frequency ratio between them.

One of the most notable, early examples of an attempt to quantify consonance/dissonance is Leonard Euler's *Gradus Suavitatus* function (Euler 1739). Gottfried Wilhelm Leibniz also approached the problem using methods remarkably similar to Tenney (Winter 2019). Not surprisingly, Hermann von Helmholtz addressed the quantification of consonance/dissonance in his seminal work *On the Sensations of Tone* (von Helmholtz 1863). A more recent example is the *harmonicity* (and corresponding *digestibility*) function of Clarence Barlow (Barlow 2001), a friend and contemporary of Tenney. I contend that Tenney's harmonic distance function synthesizes the measure of complexity into a remarkably concise function.[1]

The fundamental tenet of James Tenney's theory of harmonic distance in harmonic space is that harmonic relationships between pitches can be modeled by a multidimensional space with metric and topological properties that reflect how the human auditory apparatus perceives the relationships between pitches. In the model, as shown in Figure 3, pitches are represented by points in a multidimensional lattice, a musical graph, where the dimensions correspond to the prime factors required to specify the frequency ratios of the set of pitches with respect to a given reference pitch.

The harmonic distance between two pitches as defined by Tenney is the distance of the shortest path between the corresponding points in harmonic space. Because Tenney's harmonic space

[1] As an encouragement for any ambitious readers, a comprehensive survey and analysis of these functions, along with their subtle differences in relation to our perception of consonance and dissonance, would be an invaluable resource. I initiated such a survey in preparation for my dissertation, titled "Structural Metrics" (Winter 2010), but ultimately decided not to include the work in the final version. Juan Sebastián Lach Lau's dissertation, "Harmonic Duality" (Lach Lau 2012), includes a section on harmonic metrics that would serve as an excellent starting point.

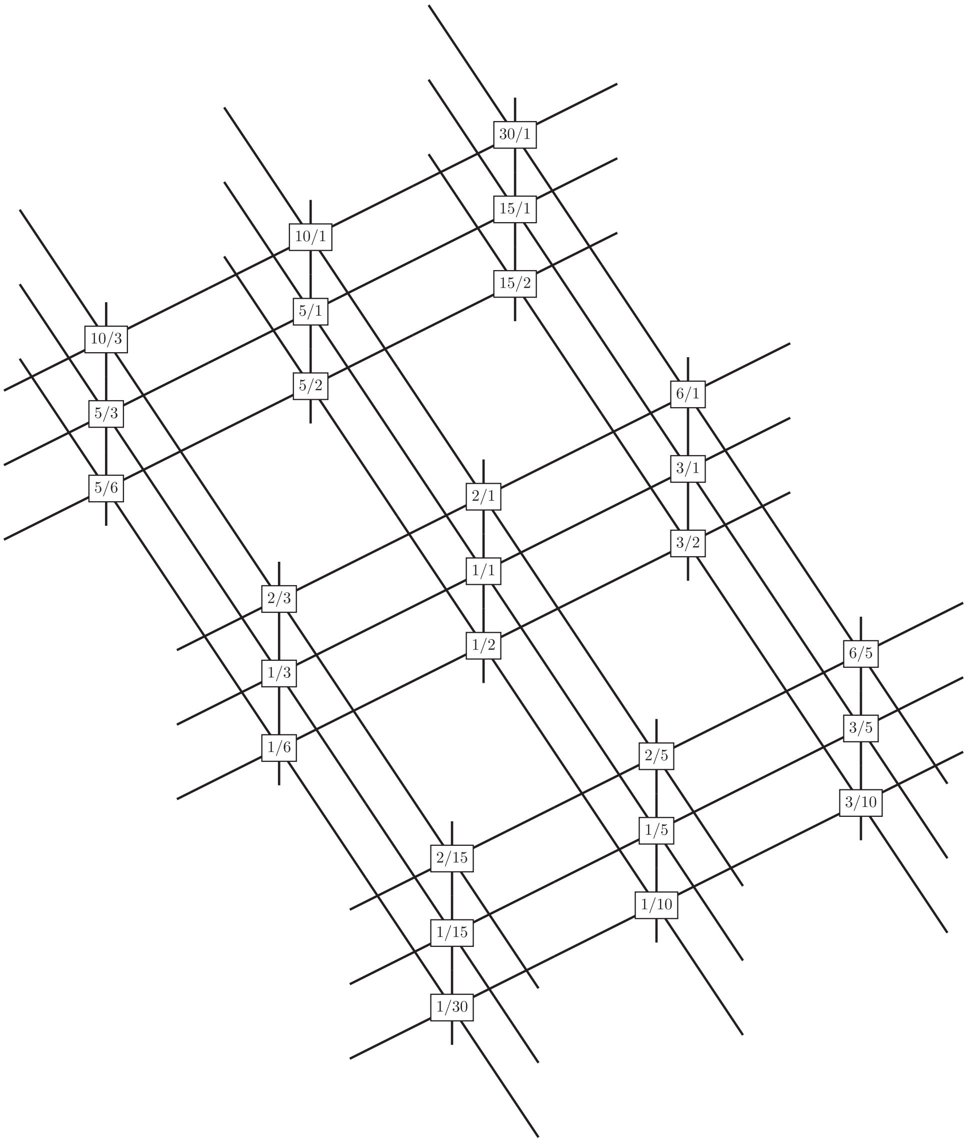


Figure 3. Example of a subset of HS_5.

is defined as a lattice, harmonic distance is a non-euclidean "city-block" metric. Tenney's mathematical formulation is beautifully elegant: $HD(a, b) = \log_2(ab)$ where a/b is a frequency ratio such that a and b are coprime. It is clear to see that harmonic distance is a concise quantification of a walk in harmonic space that weights the size of the prime factors because for $ab = 2^i 3^j 5^k \ldots$, $\log_2(ab) = i \log_2(2) + j \log_2(3) + k \log_2(5) + \ldots$.

Note that Tenney often "collapses" (or omits) the 2-dimension as it represents intervals of an octave as shown in Figure 4. Omitting the 2-dimension eliminates duplication of equivalent pitch-classes and allows higher dimensions to be more easily plotted. This collapse is somewhat deceiving because information encapsulated in Tenney's harmonic distance function – specifically movement in the 2-dimension – is lost. Despite this, as will be demonstrated below, the use of collapsed harmonic space poses both interesting challenges and solutions to musical problems.

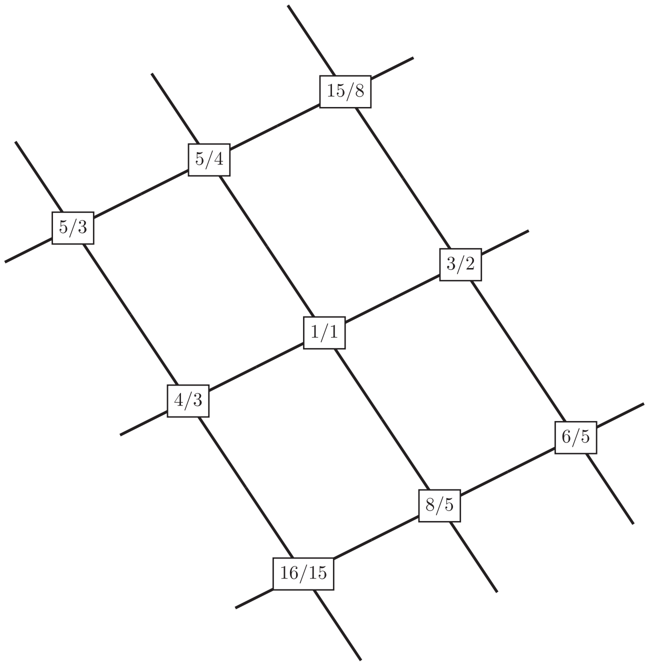


Figure 4. Example of a subset of CHS_5.

4. Modulation/movement in harmonic space and harmonic replacement

Tenney used the framework of harmonic distance in harmonic space both analytically and compositionally. His article “The structure of harmonic series aggregates” provides numerous equations to further quantify the perceived consonance/dissonance of a set of pitches beyond size two; i.e. chords (Tenney 2015d). In terms of composition, similar to the paths that Johnson found in his graphs, Tenney’s works often involve walks in harmonic space.

The methodologies Tenney implemented to move through harmonic space are a defining feature of his work. Pieces such as *Changes* for six harps tuned a sixth-tone apart and the string quartet *Arbor Vitae* are prime examples. These techniques have been extended by his students. In particular, I would like to discuss one methodology used by Larry Polansky: harmonic replacement (Polansky 2018). This is a modulation technique between two sets of pitches in just-intonation such as two harmonic series. To modulate from one set to the next, each pitch in the source set is replaced by a pitch in the destination set using two functions: a function that determines the order of pitches in the source set that will be replaced and a function that determines how the pitches will be replaced. Sometimes, these two functions are combined/interdependent. For example, one of the simplest forms of this idea that Polansky implemented was a “nearest neighbor” function in which each pitch in the source set is replaced by the nearest neighbor in pitch space of the destination set that has yet to sound. At first, the nearest neighbors are close in pitch space giving a sense of voice leading, but as less pitches remain available in the destination set, some of the later replacements can grow rather large. Polansky used this technique in various ways. In works such as *Psaltery*, *FreeHorn*, and *II – V – I*, the modulation between the source and destination sets create a blur of rather complex aggregates that result throughout the interpolation. In the *Preamble* of *For Jim, Ben, and Lou*, sections of the piece are defined by the intermediate pitch gamuts that result from the modulatory process which are then used as scales that are articulated melodically. In the following section, I explain how these techniques point towards a reexamination of a ubiquitous compositional practice: voice leading.

5. Motivation of a theory of conjunct connected sets in harmonic space

Distance between pitches is typically measured in terms of subjective height expressed in units of semitones or cents (100th of a tempered semitone). This particular concept of a musical space can be referred to as pitch (or melodic) space. However, harmonic space is not directly correlated to pitch space. For example, the perfect 5th (a frequency ratio of 3/2) is one of the closest intervals in harmonic space but relatively far (7 semitones) in pitch space. On the contrary, smaller melodic and chromatic differences/movements in pitch space are often distant in harmonic space. This gives rise to several vexing musical questions. Questions that are actually reexaminations of the traditional concept of voice leading – how individual melodic lines create and maintain harmonies in aggregate while sometimes modulating – recontextualized in the phenomenological framework of just-intonation in harmonic space. How is it possible to reconcile these two very different, well-defined measures of distance? How can one tune stepwise movement in pitch space when the relationship between two tones may actually be distant in harmonic space? How can one modulate in harmonic space with melodic voice leading?

I started exploring these questions by writing custom software[2] that maintains closely related/consonant groups of tones in harmonic space among any simultaneously sounding tones, but favors smaller steps in pitch space when one voice moves melodically. The software does this by ensuring that at any given time, the set of pitches forms a connected subgraph in harmonic space.[3]

In the resulting music, any individual part would be near impossible to play by itself. However, because the set of tones always maintains a connected subgraph, each successive tone within a part can always be tuned via a relatively simple interval in harmonic space to a tone that is already sounding in one of the other parts. Results of this process sometimes have an almost baroque, contrapuntal feel; a chromatic drift in harmonic space that is constantly modulating.

6. On the structure of conjunct connected sets in harmonic space

With this custom software, I wrote a series of pieces titled *seeds and ledgers*. Afterwards, I wanted to better understand and formalize the idea. This started with two primary questions. The first question was a practical one: how can one generate all unique chords of a given size that maintain a connected subgraph in harmonic space? This is not a trivial problem given that a chord is actually an equivalence class under transposition. That is, two chords that have the same intervallic composition between all pairwise elements, yet are transposed, are not considered distinct. The difficulty of this problem is similar to the generation and enumeration of polycubes on a grid (Aleksandrowicz and Barequet 2009), which is even harder as a polycube is also consider equivalent under rotation. The second question was how to enumerate through all chords given a set of local morphological constraints that satisfy certain rules of voice leading and counterpoint (e.g. as in Johnson's *Trio* where one note stays the same and the other two move in contrary

[2] The code repository for this software is available at: https://unboundedpress.org/code/mwinter/seeds_and_ledgers

[3] In his text, *On "Crystal Growth" in Harmonic Space* (Tenney 2015c), Tenney refers to this construction as a *compact set*. Given that a compact set has a commonly used meaning in mathematics that differs from Tenney's usage, I use the term *connected set* instead. The later use of *conjunct* connected sets refers to connected sets that have points in common.

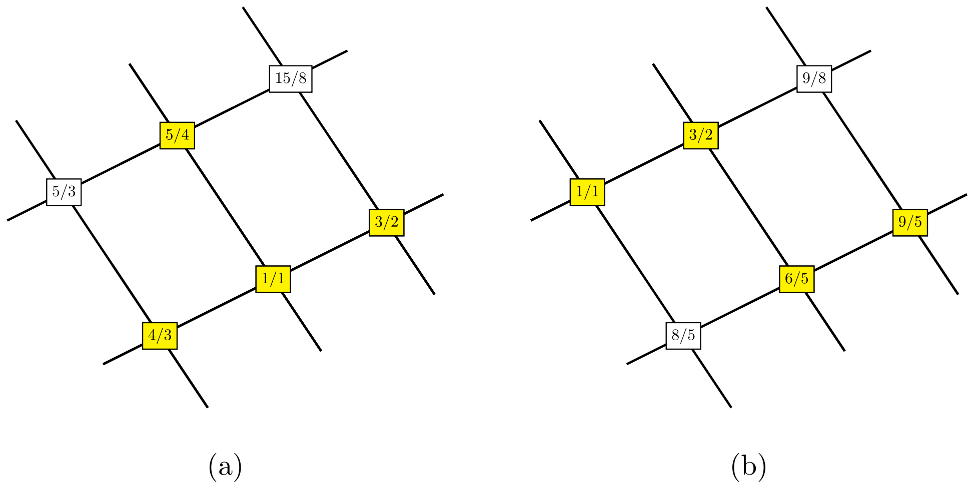


Figure 5. Two examples of connected sets of size 4 (highlighted) in CHS_5.

motion). Below, I give formal, mathematical definitions of harmonic space and connected sets and then define a set of graphs that encode desired local morphological constraints.

6.1. *Definitions*

Definition 6.1 (Frequency ratio) A *frequency ratio fr* is a ratio n/d that represents the relative frequency of a tone in comparison to a unison (1/1) such that n and d are coprime.

Definition 6.2 (Collapsed frequency ratio) A *collapsed frequency ratio* $c(fr)$ is a frequency ratio with the additional property that $1 \leq n/d < 2$. A frequency ratio can be collapsed by multiplying it successively by 2/1 if $n/d < 1$ or by 1/2 if $n/d \geq 2$ until $1 \leq n/d < 2$.

Definition 6.3 (Harmonic space) *Harmonic space* HS_l is a multi-dimensional lattice where each dimension represents a unique prime factor up to the limit l (or any finite set of primes). Each point represents a frequency ratio based on the factorization of n and d such that the p-adic valuation $v_p(n)$ and $v_p(d)$ is equivalent to the number of positive and negative steps, respectively, in each dimension p outward from a reference point (e.g. 1/1) on the lattice. An example is provided in Figure 3.

Definition 6.4 (Collapsed harmonic space) A *collapsed harmonic space* CHS_l is the same as harmonic space except the dimension representing movement by the prime 2 is omitted and each point is collapsed. An example is provided in Figure 4.

Definition 6.5 (Connected set) A *connected set* in (collapsed) harmonic space CS is a set of points (a chord) $((n/d)_1, (n/d)_2, \ldots (n/d)_k)$ that form a connected sublattice of k-points in harmonic space. Examples are provided in Figure 5(a,b).

Definition 6.6 (Transposition) A *transposition* of a connected set $T_{fr}(CS)$ is the multiplication of all elements of the set by a frequency ratio. Note that a transposition is not necessarily well-defined in a non-collapsed harmonic space because a pitch might get transposed below a 1/1 or above a 2/1. If this happens and then the pitch is recollapsed, it may result in a different connected set in CHS. This fact is discussed further in Section 7.2.1, which details the enumeration of connected sets in collapsed harmonic space. More immediately, it will become relevant in the next section when we measure the connectivity of two chords based on how many pitches they have in common. In this case, a transposition in collapsed harmonic space requires the set to be expanded into non-collapsed harmonic space. An example expansion and transposition is provided in Figure 6.

6.2. *Graph of the structure of conjunct connected sets in collapsed harmonic space*

Let $S_{j,k}(CHS_l))$ be the set of all connected sets of size j to k in collapsed harmonic space CHS_l such that at least one point is equal to 1/1 in each connected set. This represents all possible chords of CHS_l.

We define a *graph of conjunct connected sets* as $G_{a,b}(S_{j,k}(CHS_l))$ where the vertices represent the elements of $S_{j,k}(CHS_l)$ and an edge is induced between two vertices, v_1 and v_2, if the size of the symmetric difference of v_1 and some transposition of v_2 (i.e. $T_{fr}(v_2)$) plus the difference in sizes of the two sets is between a and b:

$$a \leq |v_1 \Delta T_{fr}(v_2)| + ||v_1| - |v_2|| \leq b \tag{1}$$

For example, if $a = 1$ and $b = 2$, all but one note stay the same from chord to chord if they are the same size, only one note is added or removed if the chords differ by 1 in size, and movement between chords that have a difference in size greater than 1 are not allowed. Note that these generalizations came from an original investigation where $j = k$ and $a = b = 2$, which is all connected sets of size j such that from chord to chord, only one note changes. An example graph is provided in Figure 7 followed by an example, in musical notation of a valid edge crossing in the graph under transposition (Figure 8).

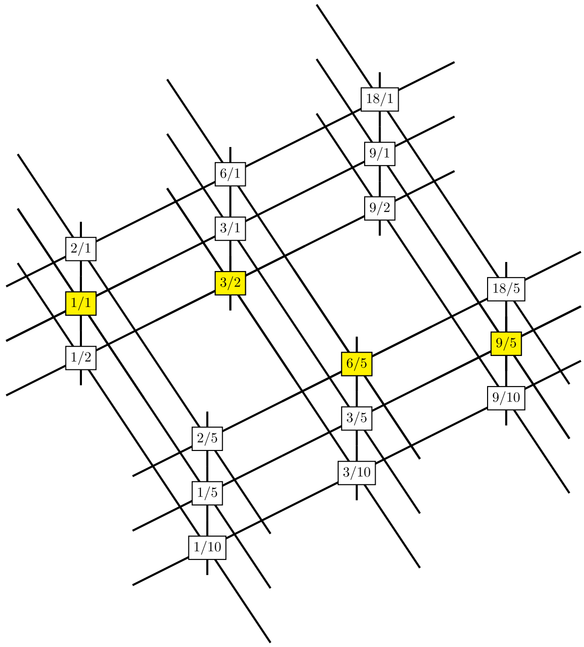
6.3. *Voice leading graphs*

The graphs in the previous section do not limit the difference in pitch space between the vertices that change from set to set. As a result these graphs are highly connected. One can also remove or weight edges based on whether or not the movement of the pitches that change between the two connected sets is within a certain melodic threshold (i.e. moves by less than a certain interval in pitch space). I refer to a graph with these added local morphological constraints as a *voice leading graph*. Let $v_1 \Delta T_{fr}(v_2) = \{fr_1, fr_2\}$, then keep or weight the edge between them if $1200(|\log_2(fr_1) - \log_2(fr_2)|)$ is less than an arbitrary limit (e.g. 200 for a whole tone). This greatly restricts the connectivity as is shown by the difference of $G_{2,2}(S_{3,3}(CHS_7))$, which is all 3 note chords in CHS_7 such that only one tone changes from chord to chord in comparison to the derived voice leading graph, where the movement of the tone that changes must be less than 200 cents (shown in Figure 9).

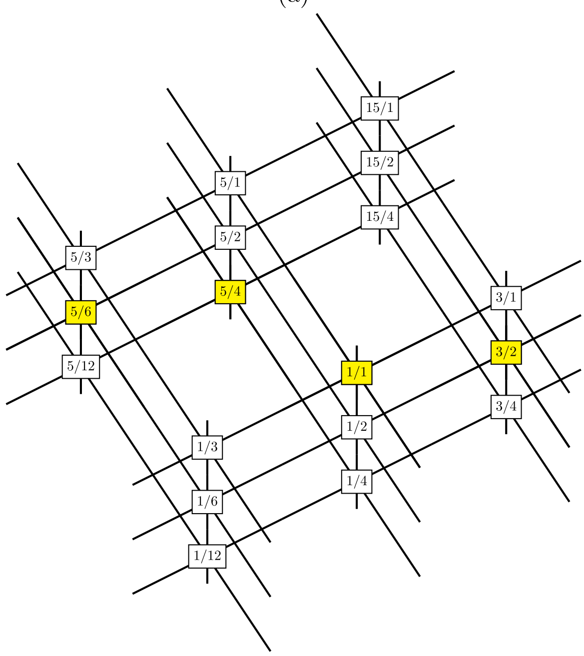
7. Conclusion

7.1. *Musical results using paths in voice leading graphs*

To date, I have written a set of string trios, *compact sets 1–3*, exploring my formalization of the structure of conjunct connected sets in harmonic space with voice leading constraints. I am currently working on another set of pieces for tunable air raid sirens that I have designed and built. These pieces are generated by finding paths in voice leading graphs using a variety of local morphological constraints. Specifically, *compact set 2* uses the same local morphological



(a)



(b)

Figure 6. Example of a connected set from Figure 5(b) expanded into non-collapsed harmonic space (a) and then transposed (b) by a frequency ratio of 5/6: minus one step in the prime 2 dimension, minus one step in the prime 3 dimension, and plus one step in the prime 5 dimension.

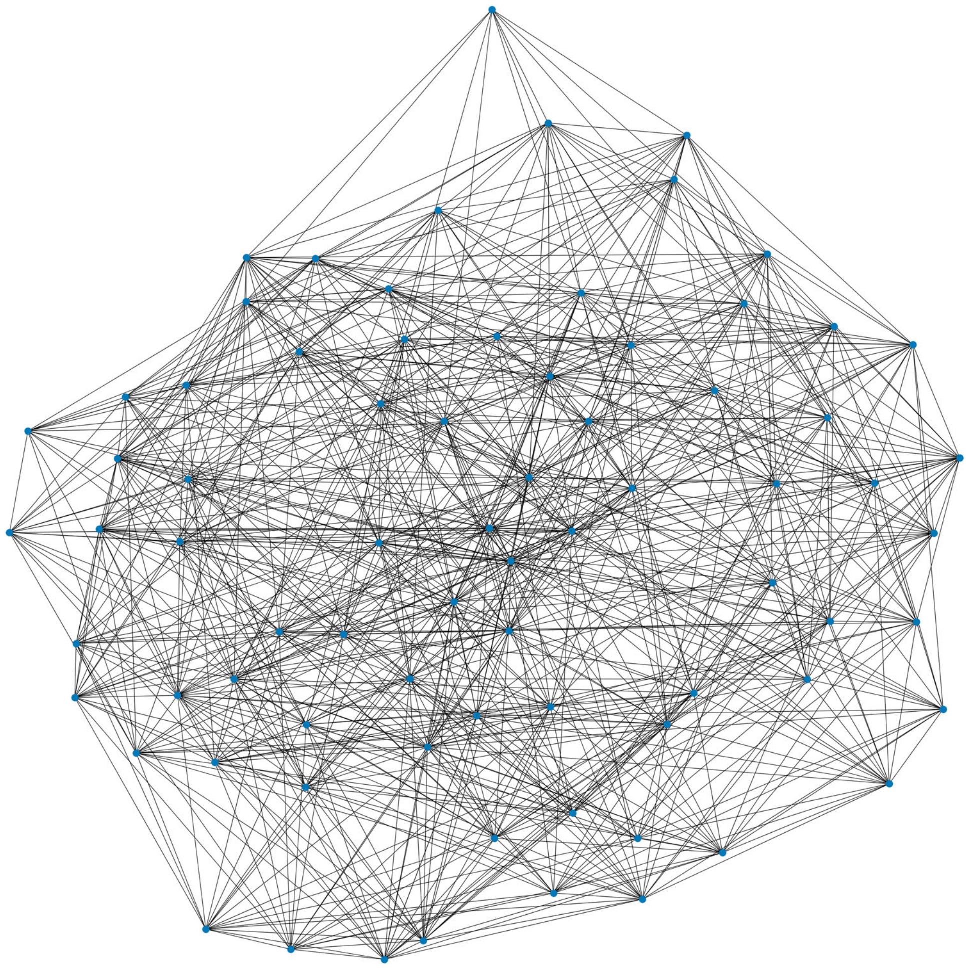


Figure 7. Example of graph $G_{2,2}(S_{4,4}(CHS_5))$.

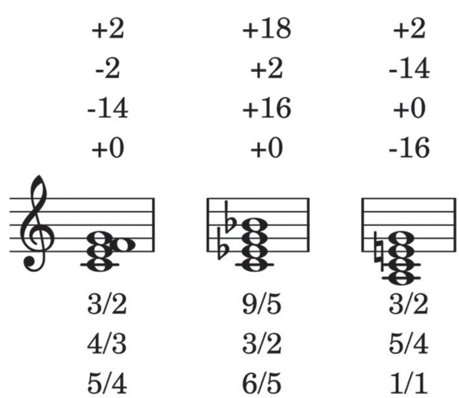
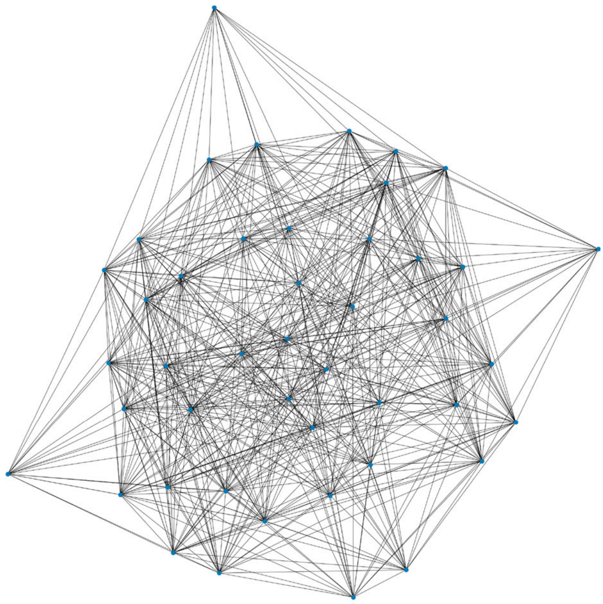
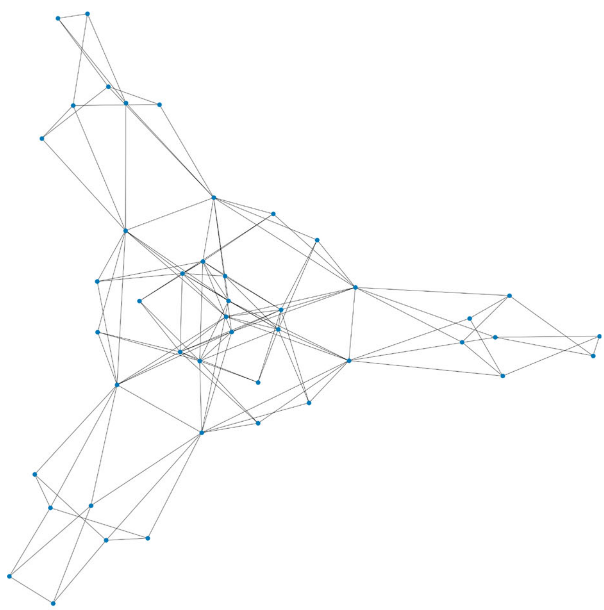


Figure 8. Chords in music notation (with frequency ratios below and cent-deviations above) of Figure 5(a,b) (equivalent to 6(a)), and 6(b), respectively, demonstrating that Figure 5(a,b) only satisfy the edge constraint in $G_{2,2}(S_{4,4}(CHS_5))$ under transposition by a frequency ratio of 5/6.



(a)



(b)

Figure 9. Example of graph $G_{2,2}(S_{3,3}(CHS_7))$ (a) and the voice leading graph derived from it such that the pitch movement of the tone that changes between chords is less than 200 cents (b).

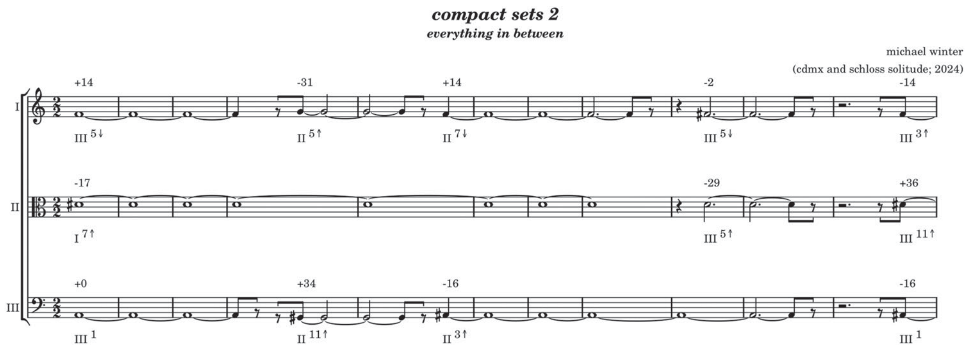


Figure 10. First system of *compact sets 2*.

constraints as in Johnson's *Trio*: that one pitch stays the same, while the other two move in contrary motion. In my piece, I limit any melodic step to within 50 and 200 cents. My original objective was to find a Hamiltonian path in this graph. As explained below, the Hamiltonicity of these graphs in general remains an open question. As an alternative of a lengthy exhaustive brute-force search for such a path, I decided to stochastically aim for an enumeration of all possible chords in a relatively short, though not necessarily Hamiltonian, path by favoring vertices that have not been crossed while allowing vertices to be recrossed in case an impasse is reached.

A score excerpt is provided in Figure 10. The notation is in what Polansky called a "paratactical" or "adaptive" tuning (Polansky 2018). While there is no fixed fundamental, each note can always be tuned via a relatively simple interval from a note that is currently sounding or has recently terminated in one of the other parts explained as follows. For the purposes of this explanation, the current note will be referred to as the *referencing note* and the pitch against which it can be tuned will be called the *reference pitch*. Each written note indicates the closest pitch in twelve-tone equal temperament with a cents-deviation (100th of a tempered semitone) provided above. A Roman numeral below a note indicates the part in which the reference pitch is. The Arabic superscript and the corresponding arrow indicate the exact interval, up or down, of the referencing note from the reference pitch as detailed below. A Roman numeral with a superscript of 1 below a referencing note means that its pitch is an octave equivalent of the reference pitch. A Roman numeral with a superscript of 3, 5, 7, 11, or 13 followed by an up arrow indicates that the pitch of the referencing note is a frequency ratio of a 3/2, 5/4, 7/4, 11/8 or 13/8, respectively, from the reference pitch if the reference pitch were transposed to the nearest octave *below* the pitch of the referencing note. A Roman numeral with a superscript of 3, 5, 7, 11, or 13 followed by a down arrow indicates that the pitch of the referencing note is a frequency ratio of a 2/3, 4/5, 4/7, 8/11 or 8/13, respectively, from the reference pitch if the reference pitch were transposed to the nearest octave *above* the pitch of the referencing note. That is, the down arrow is the inversion of an up arrow and it could alternatively be understood that the referencing note is a frequency ratio of 4/3, 8/5, 8/7, 16/11, or 16/13, respectively, from the reference pitch if the reference pitch were transposed to the nearest octave *below* the pitch of the referencing note.

7.2. *Open questions in graph theory and computational complexity*

As a result of my explorations, several questions arise with respect to graph theory and computational complexity.

7.2.1. *Generating all unique connected sets in a given collapsed harmonic space*

I have developed a recursive algorithm that will generate all unique connected sets in a given collapsed harmonic space reasonably fast by “growing” the compact sets/chords.[4] The Python code is provided in Figure 11. Whether or not the algorithm is optimal (or near-optimal) remains an open question.

As previously mentioned, the problem of generating all unique sets in harmonic space is related to the generation of polycubes (Aleksandrowicz and Barequet 2009); however, with certain constraints relaxed. Polycubes are the same as connected subgraphs on a grid under equivalence of translation and rotation. Testing for rotation increases the computational complexity, in terms of computation time, orders of magnitude. However, generating connected subgraphs in non-collapsed harmonic space does not require checking for equivalence under rotation, only translation (which is the same as transposition). I focus on connected sets in collapsed harmonic space precisely because it further relaxes the constraints of the problem. For the set of all connected sets of a given size in collapsed harmonic space that share a common point, each connected set is a unique chord and any translation of a connected set that does not include that common point will have the same intervallic content as a connected set that does include the common point. As such, translation in collapsed harmonic space is not the same as transposition, it is more like a homomorphism to another connected set. Transposing the frequency ratios of a set and then re-collapsing them has a similar effect. That is why generating all compact sets in collapsed harmonic space is much more efficient as one can grow all connected sets up to a given size from a common point (which can be, for example, the 1/1) without checking for rotation or translation.

7.2.2. *Generating graphs of conjunct connected sets and voice leading graphs*

I have not attempted, in any way, to optimize the generation of graphs of conjunct connected sets and voice leading graphs. The method I implemented in *compact sets 1–3* is extremely computationally expensive. The algorithm has to test each pair of sets to see if they satisfy the local morphological constraints. Further, it also tests all possible transpositions of the destination chord that share at least one note in common with the source chord. A positive answer to the following question would optimize the graph generation. Given two connected sets, is there an efficient way to determine if a transposition exists such that Equation 1 in Section 6.2 is satisfied? Given that the algorithm to generate connected sets is recursive, I believe it might also be possible to build these graphs while generating the connected sets themselves by keeping track of how distant the chords are from one another in terms of symmetric difference.

7.2.3. *Hamiltonicity*

Finally, as I have already touched upon, a fundamental question related to graph theory is whether graphs of conjunct connected sets and voice leading graphs are Hamiltonian and if so, under what conditions. The number of dimensions of the harmonic space plays a role and as demonstrated by Figure 9, even simple voice leading constraints greatly reduce connectivity.

[4] This algorithm is a modification to the algorithm provided at the following url: https://stackoverflow.com/questions/75727217.

```python
def connected_sets(hs_dimensions, min_chord_size, max_chord_size):

    # return numeric value of frequency ratio of pitch from its array
    def hs_array_to_fr(hs_array):
        prod = 1
        for d in range(len(hs_dimensions)):
            prod *= pow(hs_dimensions[d], hs_array[d])
        return prod

    # this function inserts the value in the 2-dimension so that the pitch is between 1 and 2
    def wrap_pitch(chord):
        wrapped_pitch = list(chord)
        while (fr := hs_array_to_fr(wrapped_pitch)) < 1 or fr >= 2:
            wrapped_pitch[0] += 1 if fr < 1 else -1
        return tuple(wrapped_pitch)

    # runs wrap_pitch on a chord
    def wrap_chord(chord):
        return tuple(wrap_pitch(p) for p in chord)

    # return all vertices in the lattice connected to the input vertex
    def branch(vertex):
        # initialize empty set
        branches = set()
        # iterate through the branches
        for i,d in enumerate(hs_dimensions):
            # do not run on the 2-dimension as this is for collapsed harmonic space
            if d != 2:
                # generate the branches +1 or -1 from the current dimension
                for alt in [-1, 1]:
                    branch = (*(v:=vertex)[:i], v[i] + alt, *v[(i + 1):])
                    branches.add(branch)
        return branches

    # recursive function to generate all connected sets
    def grow(chord, connected, visited):
        # output chord if it is within the chord size limits
        if len(chord) >= min_chord_size and len(chord) <= max_chord_size:
            wrapped_chord = sorted(wrap_chord(chord), key=hs_array_to_fr)
            yield tuple(wrapped_chord)
        # this keeps growing the chord
        if len(chord) < max_chord_size:
            visited = set(visited)
            # iterate through all the connected vertices from previous recursion
            for b in connected:
                # check to see if the vertex has already been visited
                if b not in visited:
                    # extend the chord if the vertex has not been visited
                    extended_chord = (*chord, b)
                    # branch from this vertex and add them to the list of connected vertices
                    new_connected = connected | branch(b)
                    # add new vertex to the list of visited vertices
                    visited.add(b)
                    # recurse
                    yield from grow(extended_chord, new_connected, visited)

    # initialization
    root = tuple(0 for d in hs_dimensions) #initialize the 1/1, e.g. (0, 0, 0, 0), which is the common point
    connected = branch(root) # initialize branches at first level
    visited = set((root,)) # initialize the set of traversed vertices
    yield from grow((root,), connected, visited) # start recursion

# example of all chords of size 4 in CHS_7
harmonic_space_dimensions = (2, 3, 5, 7)
chord_set = list(connected_sets(harmonic_space_dimensions, 4, 4))
```

Figure 11. Python code that generates the set of all connected sets. Note that a connected set is represented as a tuple of tuples, where each pitch in a chord is described by steps in harmonic space in each prime dimension. This representation of pitch can easily be converted into a frequency ratio using the `hs_array_to_fr` function.

7.2.4. *Final remarks*

Johnson sometimes felt that his pieces failed if the listener was unable to hear the logic. I take a more open stance that I call the *incalculability of concept-to-percept transparency*. This idea, first posited in my dissertation (Winter 2010), states that one cannot assume the degree to which a listener will be able to understand and articulate the conceptual basis of the piece after they perceive/experience it. However, Johnson's rigorous hope was the result of his exacting mind; a most inspiring quality. One aspect of his music I feel comfortable articulating, which is also how I describe the work of Tenney and Polansky – despite each of these composers having such distinct aesthetics – is that when you listen to the music, it is so clear that *something* is happening. You can hear that there is a logic even if you cannot articulate exactly what it is.

Borrowing from Iannis Xenakis's concept of *outside-time musical structures* (Xenakis 1992), Tenney's theory of harmonic space is fundamentally a theory of perception *out-of-time*. How one moves in this space is a formal compositional consideration: that is, the temporal organization of musical material into musical morphologies. Tenney distinguishes these as *structure* and *shape*, respectively, which together comprise musical form (Tenney 2015b).

The theory of the structure of conjunct connected sets in harmonic space applies methodologies to create musical morphologies using techniques extended from Johnson and Polansky into harmonic space, a musical structure defined by Tenney. The fact that some of my early experiments have resulted in sequences that bear similarity to chordal sequences in the baroque and classical periods suggests that it could be implemented as a theoretical basis for functional tonal harmony. Such an application warrants further investigation. Regardless, one thing I can say with certainty is that this theory is a direct product of the combined influence of Tenney, Polansky and Johnson. Not surprisingly, these are three people who had an insatiable hunger for understanding music and sharing their musical ideas, rather than guarding them closely in secret; ever interested in the evolution of those musical ideas.

Acknowledgments

I would like to thank Lucie Nezri and Jarkko Kari for the support in the preliminary formation of these ideas. I would also like to thank the reviewers and editors, whose comments and suggestions greatly improved the text.

Disclosure statement

No potential conflict of interest was reported by the author.

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